Sensitivity of Initially Stressed Un-symmetric Micro-Layered Plate in Large Deflection due to Lateral Load

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ABSTRACT

The sensitivity of an isotropic un-symmetric layered micro-layered plate under initial tension is formulated analytically. The approach which follows von Karman large deflection plate theory for the case of a un-symmetrically layered isotropic plate was developed, first, followed by neglecting the arising nonlinear terms to have a linear consideration upon the posed problem. The arisen modified Bessel equation for the lateral slope can be solved by the use of modified Bessel functions with arguments defined via a modified tension parameter accounting for stiffness due to un-symmetry. The solutions for geometrical responses are further manipulated to obtain an analytical expression for the mechanical sensitivity for a typical layered micro-sensing plate, using the re-occurrence relationships for integration and differentiation for the modified Bessel functions.

The results for a slightly un-symmetric plate are compared to those of a singled layered case available in literature to validate the present approach. The radial variations of the geometrical responses are found to correlate very well with which given by Sheplak and Dugundji [1], throughout the range of the employed initial tension. In additions, the effects of initial tension, lateral load, as well as the deviation in layer moduli are thoroughly investigated, for typical un-symmetrically

Keywords：Un-symmetric layered plate, Sensitivity, Initial tension, von-Karman Plate theory, Modified Bessel functions.

1. INTRODUCTION

Initial tension often arises in a typical micro-fabrication process. It has been well recognized that the magnitude of initial tension could be high enough to cause a serious degradation in structural performance such as the deflection-based pressure sensitivity. For miniaturized devices such as poly-silicon-based pressure sensors and accelerometers, they are commonly fabricated in an un-symmetrically layered configuration and large deflection condition is often encountered in application. In this case, classical plate theory based on Kirchhoff assumption is no longer applicable and thus requiring a more precise theoretical consideration to predict the behavior of the micro-sensing structure. The coupled effect due to pretension and un-symmetry upon the relevant structural responses is equally important. In particular, it may be worthy of note about the associated mechanical sensitivity for such typical sensing devices undergoing a large deflection condition.

The recent close-related works considering large deflection of sensor plates under initial tension are due to Voorthuyzen & Bergveld [2], Sheplak and Dugundji [1], and Su et al. [3]. These studies employed either analytical approach or numerical finite difference method incorporated with an iteration scheme to solve for the corresponding simplified linear problem and the associated nonlinear problem, for either an isotropic micro-plate with or without a center boss. Nevertheless, the practical problem of mechanical sensitivity for the sensing device seemed to be ignored until an analytical formulation was presented by Saini et al. [4]. In this study, the linear solution for the geometrical responses of a single-layered sensor plate was further manipulated, providing an analytical formulation for scaling the mechanical sensitivity of the sensor. Unfortunately, no detailed information regarding material properties and geometry of the plate were given, despite an insight of the effect of initial tension upon the mechanical sensitivity was indeed presented.

For an un-symmetrically layered composite sensing device, the coupled effect due to pretension and un-symmetry upon the relevant structural responses can be very important. In particular, it may be worthy of note about the associated mechanical sensitivity for such typical sensing devices undergoing a large deflection condition. To this end, the sensitivity of an isotropic un-symmetric layered micro-layered plate under initial tension (Figures 1) is formulated analytically. The approach which follows von Karman large deflection plate theory for the case of a un-symmetrically layered isotropic plate was developed, first, followed by neglecting the arising nonlinear terms to have a linear consideration upon the posed problem. The simplified equation set leads to a modified Bessel equation

for the lateral slope with solution expressible in terms of modified Bessel functions with arguments defined via a modified tension parameter accounting for stiffness due to un-symmetry. The solutions for geometrical responses are developed and further manipulated to obtain an analytical expression for the mechanical sensitivity for a typical layered micro-sensing plate, using the re-occurrence relationships for integration and differentiation for the modified Bessel functions. Some parametric study regarding the effect of layer un-symmetry and initial tension upon the geometrical and stress responses, as well as the mechanical sensitivity of the un-symmetric plate will be conducted.

2. PHYSICAL PROBLEM AND SOLUTION

An un-symmetrically layered circular plate clamped all around is considered. It is subjected to a uniform pretension, N_0, and a uniform lateral pressure , P_0, as shown (Fig. 1 and 2). The governing equations which follow the von Karman's plate theory of large deflection. can be written in terms of radial slope, $w,_r$, and in-plane force resultants, N_r and N_θ, as well as the moment resultants, M_r and M_θ, i. e.,

$$\begin{cases}(rN_r),_r - N_\theta = 0, \\ (rQ_r),_r + (rN_r w,_r),_r = -qr, \\ (rM_r),_r - M_\theta - rQ_r = 0;\end{cases} \qquad (1)$$

where Q_r is the transverse shear force resultant. In additions, the force resultants, Ns, and moment resultants, Ms, are defined through the laminate constitutive laws for a layered isotropic plate. By using the strain-displacement relations, integrated form of the second equilibrium equation, and the laminate constitutive laws, the second and the third equations can be rewritten in terms of w', N_r and N_θ. Focusing the case of unique Poisson's ratio for the layers of the plate, and expressing N_θ in terms of N_r, via the use of equation (1), these 3 nonlinear equations can further be recast and merged into 2 equations for the lateral slope, $\theta = w'$, and N_r such that,

$$B_{A1} r^2 N_r''' + B_{A2} r N_r'' + B_{A3} N_r' + T_a \frac{r^2\theta'' + r\theta' - \theta}{r^2} + \frac{B_-}{2r^2}\theta^2 + B_r\theta\theta' + N_r\theta + \frac{qr}{2} = 0 \qquad (2)$$

$$A_r^{-1} r^2 N_r'' + 3A_r^{-1} r N_r' + \theta^2/2 = 0 \qquad (3)$$

where $B_{A1} = B_r A_r^{-1}, B_{A2} = B_r A_t^{-1} + 6B_{A1}, B_{A3} = 3B_r A_t^{-1} + 6B_{A1}$; $T_a = B_r C_\beta - D_r$, $B_- = B_r - B_t$; $w' = dw/dr$, $C_\beta = A_r^{-1} B_r + A_t^{-1} B_t$, $A_u = A_t^{1} - A_r^{1}$, In additions, As, Bs, and Ds are the elements of the extensional, the coupling, and the bending stiffness matrix of the un-symmetrically-layered isotropic plate; and $A_r^{-1} = [A]_{11}^{-1}$, $A_t^{-1} = [A]_{12}^{-1}$, respectively. Employing similar non-dimensional scheme as which used by Sheplak and Dungundji [1], i. e.,

$$\xi = r/a,\ W = w/h,\ U = u/h; \qquad \Theta = \frac{dW}{d\xi} = \frac{a}{h}\frac{dw}{dr};$$

$$\Psi = \frac{d\Theta}{d\xi} = \frac{a^2}{h}\frac{d^2 w}{dr^2} = \dot{\Theta}; \qquad P = q_0 a^4 / E_1 h^4;$$

$$[S_r, S_\theta] = [\hat{N}_r, \hat{N}_\theta]\, a^2 / D_r; \quad k = \sqrt{N_0 a^2 / D_r}$$

equations (14) and (15) can further be manipulated to read,

$$T_a\left(\xi^2\ddot{\gamma} + \xi\dot{\gamma} - \gamma\right) + \frac{B_-}{2a^2}\gamma^2 + B_r\xi^2\gamma\dot{\gamma} + D_r\left[B_{A1}\xi^4\dddot{S}_r + B_{A2}\xi^3\ddot{S}_r + B_{A3}\xi^2\dot{S}_r + \xi^2\left(S_r + k^2\right)\gamma\right] + P_n\xi^3 = 0 \quad (4)$$

$$D_r A_r^{-1}\left(\xi^3\ddot{S}_r + 3\xi^2\dot{S}_r\right) + \frac{1}{2}\gamma^2\xi = 0 \qquad (5)$$

where $\gamma = h\Theta$, $\dot{\gamma} = d\gamma/d\xi$, …etc; and $P_n = qa^4/2$. U, W, θ, and ψ are the non-dimensional radial displacement, lateral deflection, lateral slope and curvature respectively; and ξ, P and k are the non-dimensional radial coordinate, lateral pressure, and tension parameter individually. Among them, the dimensionless pressure (P) and pretension parameter (k) are defined such that,

$$P = \frac{p_0 a^4}{D_l h}, \quad k = \sqrt{\frac{N_0 a^2}{D_l}}$$

These equations are subject to the following boundary conditions of the problem, i. e., (i) For $\xi = 0$: $\theta = 0$, $S_r = S_\theta$; and (ii) For $\xi = 1$: $\theta = 0$, $U = 0$.

For the case of small deflection as tension parameter, k, varies from 0 to infinity, all the nonlinear terms in equations (16) and (17) can be neglected, yielding a presumed constant for the non-dimensional radial force resultant, S_r; and a linear differential equation for $\gamma = h\Theta$,

$$\xi^2\ddot{\gamma} + \xi\dot{\gamma} - \left(1 + k_D^2 \cdot \xi^2\right)\gamma = -P_T\xi^3 \qquad (6)$$

where $k_D^2 = k^2 D_T$, $D_T = D_r / (D_r - B_r C_\beta)$, and $P_T = P_n / T_a$. The solution for γ includes a very straight forward particular solution and a homogeneous part expressible in terms of Modified Bessel functions of the first kind and the

second kind, $I_1(\eta)$ and $K_1(\eta)$ $(\eta = k_D\xi)$ [5], i. e., .

$$\gamma(\xi) = h\Theta(\xi) = \frac{P_T}{k_D^2}\left[\xi - \frac{I_1(\eta)}{I_1(k_D)}\right] \qquad (7)$$

The lateral deflection and radial curvature are obtained by integrating and differentiating (20), respectively, i. e.,

$$W(\xi) = \frac{P_T}{k_D^2 h}\left[\frac{\xi^2 - 1}{2} - \frac{I_0(\eta) - I_0(k_D)}{k_D I_1(k_D)}\right] \qquad (8)$$

$$\Psi(\xi) = \frac{P_T}{hk_D^2}\left\{1 - \frac{k_D I_0(\eta)}{I_1(k_D)} + \frac{I_1(\eta)}{\xi I_1(k_D)}\right\} \qquad (9)$$

By deriving the strain components at any specific depth in the layered plate, the corresponding mechanical sensitivity of the sensor plate (scaled as the ratio between the arisen stress and the applied stress [4]) can further be formulated using the reduced stiffness relationships for the layers and the re-occurrence relationships among the modified Bessel functions [5].

3. NUMERICAL REMARKS

For the purpose of demonstration, 2-layered un-symmetric plates are implemented with various thicknesses for the layers in stacking and a range of ratios of Young's moduli. The considered ratios between the layer thickness are t_1/t_2=1.0, 0.5, 0.25; and those between the Young's moduli of the layers are taken to be, E_1/E_2=10.0, 5.0, 2.0, for studying the effect of mismatch in modulus. Poisson's ratios for all the layers are assumed to be the same as which for a typical silicon-nitride, i. e., v_1=v_2=0.27 [1]. In additions, the same range of initial tension as which used by Sheplak and Dugundji [1] is adopted here, i. e., k=1, 5, 10, 20, 50. It should be noted, however, although the normalized solutions following the present current approach appear to be the same as which given by Sheplak and Dugundji [1], provided the present k_D replaces k in [3], the values for tension parameter are taken for k here, instead of k_D, to have an insight of the effect of coupling stiffness due to the layer un-symmetry. The obtained solutions include the normalized geometrical responses of the plates, i. e., the lateral deflection, slope, and curvature, versus the dimensionless radial coordinate.

A 2-layered plate with E_1/E_2=1.2 and a ratio of t_1/t_2=0.25 between the layer thicknesses is considered, first, simulating a slightly un-symmetric condition. The radial variation of the lateral slope is presented in Fig. 2. It is seen, the results correlates very well with which given by Sheplak and Dugundji [1], throughout the range of the employed initial tension.

To study the effect of initial tension upon the mechanical sensitivity of the un-symmetric layered plate, the presented approach is further implemented with the layer materials and configurations of Zhou et al. [6]. With a slightly un-symmetric consideration for a 3-layered plate such that the Young's moduli are given as, E_1, E_2 and E_3=1.7, 1.6, 1.6 (10^{11} Pa) the Poisson's ratios to be all taken to be 0.27, and the layer thickness are 0.4, 0.2, and 0.4 (10^{-6} m), the normalized solutions for the radial stress on the top (σ_{rt}) and the bottom surface (σ_{rb}), as well those for the circumferential stress on the top (σ_{tt}) and the bottom surfaces (σ_{tb}) are presented in Figures 3 – 6, respectively. Numerically, the solutions for the radial stress on the top surface are found to be identical to the results calculated by using the formulation presented in Saini et al. [5], provided the present problem configuration is considered.

4. REFERENCES

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5. ACKNOWLEDGEMENT

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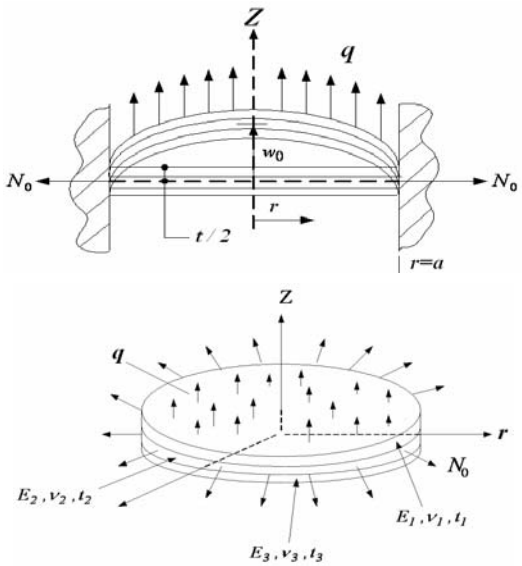

Fig. 1 Clamped and Laterally Loaded Un-Symmetrically Layered Plate under Initial Tension

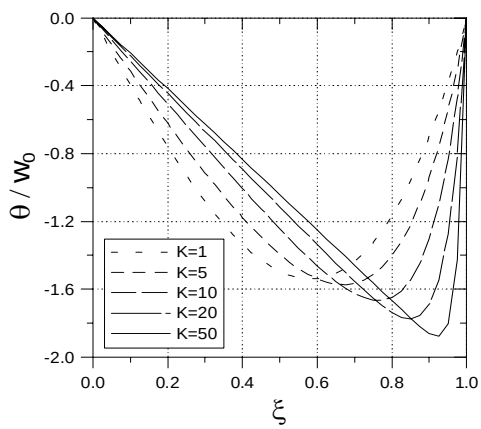

Fig. 2. Slope of 2-layered Plate, E_1/E_2=1.2, t_1/t_2=0.25

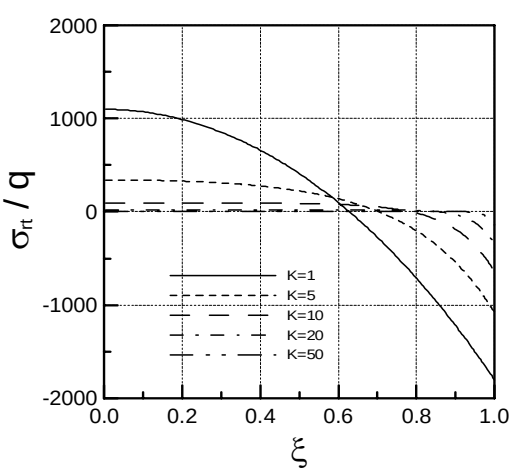

Fig. 3 Sensitivity, σ_{rt}/q, on top surface of a 3-layer Plate

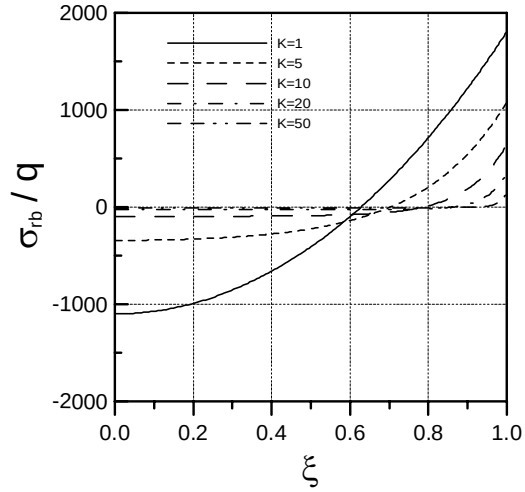

Fig. 4 Sensitivity, σ_{rb}/q, on top surface of a 3-layer Plate

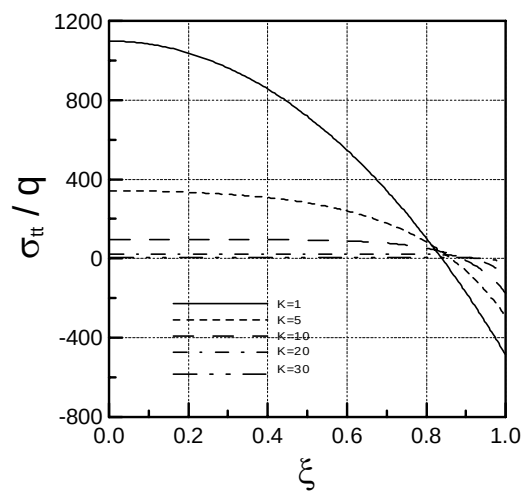

Fig. 5 Sensitivity, σ_{tt}/q, on top surface of a 3-layer Plate

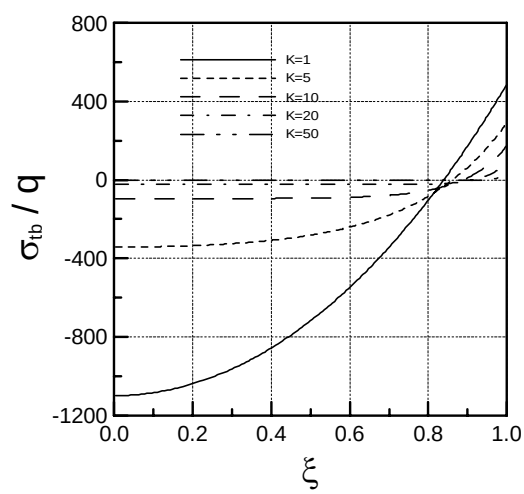

Fig. 6 Sensitivity, σ_{tb}/q, on top surface of a 3-layer Plate