Electromechanical buckling of a pre-stressed layer bonded to an elastic foundation

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ABSTRACT

The ElectroMechanical Buckling (*EMB*) of a pre-stressed layer bonded to an elastic foundation is analyzed. A new analytic solution of the mechanical post-buckling is presented. In addition, it is shown that electrostatic forces can precipitately instigate buckling even when the pre-stress is lower than the critical value that allows mechanical buckling.

Keywords: buckling, electrostatic instability, electromechanical buckling.

1 INTRODUCTION

Mechanical buckling is a well-known phenomenon that occurs in thin elastic structures which are subjected to compressive loads. Mechanical buckling develops only if the compressive loads are larger than a critical value. In most structures, reduction of a post-critical compressive load to a sub-critical level will eliminate the buckling deformation. In thin sheet-like elastic solids that are bonded to an elastic foundation, a compressive stress can cause a dense occurrence of buckling flexures.

A different kind of instability that may develop in deformable solids is due to the application of electrostatic forces. The inherent nonlinearity of electrostatic attraction forces can cause the well known pull-in electromechanical instability.

This work investigates the electromechanical response of an electrically conducting, pre-stressed elastic layer, bonded to an elastic foundation, which is subjected to both electrostatic forces and compressive loads. The analysis suggests that for a sub-critical compressive stress, buckling can be precipitately instigated by application of electrostatic forces that destabilize the structure. Elimination of these forces eliminates the buckling deformations.

The presented analysis considers a one dimensional model of the problem. Figure 1 describes an electrically conductive, pre-stressed thin layer bonded to a dielectric elastic foundation that is fixed to a conductive solid substrate.

2 MECHANICAL BUCKLING

The governing equation of the problem is given in the following normalized form

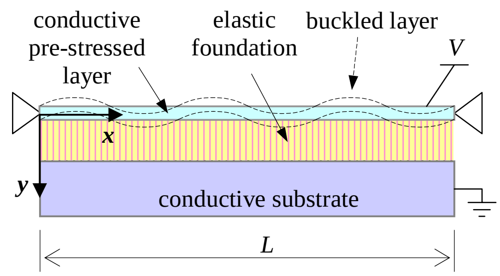

Figure 1: A pre-stressed conductive layer bonded over a dielectric elastic foundation.

$$\frac{1}{(2\pi)^4}\frac{d^4\tilde{y}}{d\tilde{x}^4}+2\beta\frac{1}{(2\pi)^2}\frac{d^2\tilde{y}}{d\tilde{x}^2} -\frac{1}{\alpha}\left[\int_0^{\alpha}\frac{1}{2}\left(\frac{d\tilde{y}}{d\tilde{x}}\right)^2 d\tilde{x}\right]\frac{1}{(2\pi)^2}\frac{d^2\tilde{y}}{d\tilde{x}^2}+\tilde{y}=\tilde{q} \qquad (1)$$

where

$$\tilde{x}=\frac{x}{\Lambda_{cr}},\qquad \tilde{y}=\frac{\sqrt{S}\,y}{\Lambda_{cr}},\qquad \beta=\frac{\sigma}{\sigma_{cr}},\qquad \tilde{q}=\frac{\sqrt{S}\,q}{\Lambda_{cr}k_f},$$

$$S=\frac{EA}{\sqrt{k_f}\sqrt{E^*I}},\quad \sigma_{cr}=-\frac{2\sqrt{k_f E^* I}}{A},\quad \Lambda_{cr}=2\pi\left(\frac{E^*I}{k_f}\right)^{\frac{1}{4}}$$

Here β is the load parameter, S is a non-dimensional stiffness ratio that governs the postbuckling configuration, A, and I are the cross-section area and second moment, per unit width, E and $E^*=E/(1-\nu)$ are Young's modulus and the effective modulus in bending, ν is Poisson's ratio, k_f is the elastic foundation stiffness, and q is a distributed load. The physical meaning of the parameters α, σ_{cr}, and Λ_{cr} is revealed in the following. Throughout section 2 it is assumed that $\tilde{q}=0$.

2.1 Critical State

To extract the critical state (i.e., the verge of buckling) the linear form of the equation is obtained by omitting the third term (with the square brackets), that accounts for axial strains due to axial variations in vertical deflection. It is postulated that the buckling deformation is of the form

$$\tilde{y} = B\sin(2\pi\,\tilde{x}/\alpha) \quad (2)$$

Here B is the amplitude of the deformation, and the variable α is the normalized wavelength. Later (in 2.2-3), the buckling problem will have to be solved numerically. Since an infinite layer can not be modeled numerically, periodic solutions of layers with finite length will be considered. The specific layer length that is associated with a true period of the solution of an infinite layer, will be identified using energy considerations. In this respect, α in (2) ensures a periodic solution for a layer of length $L = \alpha\Lambda_{cr}$.

Substituting this postulated function in to the linear form of (1) yields,

$$B\frac{\sin(2\pi\,\tilde{x}/\alpha)}{\alpha^4}\left(\alpha^4 - 2\beta\alpha^2 + 1\right) = 0 \quad (3)$$

The nontrivial solution of this equation is

$$\alpha = \sqrt{\beta \pm \sqrt{\beta^2 - 1}} \quad (4)$$

α is real only if $\beta \geq 1$ and therefore the critical state of the system is β_{cr} =1, α_{cr} =1. Accordingly, it is deduced that Λ_{cr} and σ_{cr} are the critical wavelength and critical pre-stress, respectively, at which buckling occurs in an infinite layer. A form of this result was previously derived by Hetényi [1].

Figure 2 illustrates the critical states for layers with various lengths $L = \alpha\Lambda_{cr}$. However, for a sufficiently long layer (α>1) two deformation waves with wavelength α/2 may develop. For a layer that is longer still, a third mode with wavelength α/3 may also develop (and so on).

2.2 Post-Buckling State

The post-buckling state is governed by the nonlinear equation (1). Due to this nonlinearity, the governing equation is often solved using approximation methods such as approximate analytical solutions (e.g., Rayleigh-Ritz [2]) or numerical solutions (e.g., finite differences).

In this work we present an exact analytical solution of the nonlinear equation. To this end we postulate a solution of the form (2). Substituting this postulated deformation in (1) yields

$$B\sin(2\pi\tilde{x}/\alpha)\frac{1}{\alpha^4}\left[\alpha^4 - 2\alpha^2\beta + B^2\pi^2 + 1\right] = 0 \quad (5)$$

The nontrivial solution of the above equation is given by $B = (2\beta\alpha^2 - \alpha^4 - 1)^{1/2}/\pi$, and therefore, the postbuckling solution is

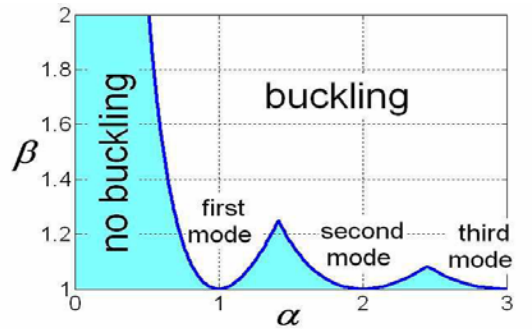

Figure 2: Critical stability of purely mechanical buckling.

$$\tilde{y} = \frac{\sqrt{2\beta\alpha^2 - \alpha^4 - 1}}{\pi}\sin(2\pi\tilde{x}/\alpha) \quad (6)$$

The actual normalized wavelength associated with a given value of β, minimizes the total strain energy (per unit length) U=U_B+U_R+U_{EF} of the system. The total strain energy is the addition of three components associated with bending deformation of the layer (U_B), axial deformation of the layer (U_R), and deformation of the elastic foundation (U_{EF}). Normalizing the strain energy components by the (axial) strain energy at the verge of buckling (U_{cr}=$\sigma_{cr}^2A/2E$), yields

$$\tilde{U}_B = \frac{1}{16\pi^2}\frac{1}{\alpha}\int_0^\alpha\left(\frac{d^2\tilde{y}}{d\tilde{x}^2}\right)^2 d\tilde{x} \quad (7a)$$

$$\tilde{U}_A = \frac{1}{\alpha}\int_0^\alpha\left(\beta - \frac{1}{2\alpha}\int_0^\alpha\frac{1}{2}\left(\frac{d\tilde{y}}{d\tilde{x}}\right)^2 d\tilde{x}\right)^2 d\tilde{x} \quad (7b)$$

$$\tilde{U}_{EF} = \frac{\pi^2}{\alpha}\int_0^\alpha\tilde{y}^2 d\tilde{x} \quad (7c)$$

Substituting (6) into (7) yields

$$\tilde{U} = \tilde{U}_B + \tilde{U}_A + \tilde{U}_{EF} = \frac{\alpha^4+1}{4\alpha^4}\left(4\alpha^2\beta - \alpha^4 - 1)\right) \quad (8)$$

For arbitrary loads $\beta \geq 1$, the normalized total strain energy has a minimum at $\alpha = 1$ (the other roots of $d\tilde{U}/d\alpha = 0$ are non physical).

Accordingly, it is deduced that the postbuckling wavelength is identical to the critical wavelength, and that the postbuckling deformation is given by

$$\tilde{y} = \frac{\sqrt{2\beta - 2}}{\pi}\sin(2\pi\tilde{x}) \quad (9)$$

Figure 3 illustrates the total strain energy of finite layers with various normalized wavelength α. For a given pre-stress β, the minimal energy is associated with $\alpha = 1$ (and the integer multiplications $\alpha = 2,3,...$).

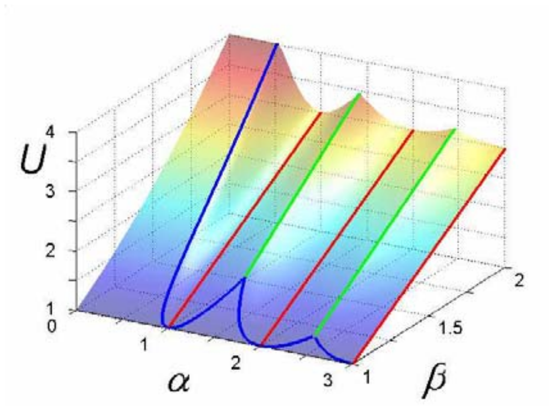

Figure 3: Total strain energy $\boldsymbol{U}$ of the pre-stressed layer. For a given pre-stress $\boldsymbol{\beta}$, the wavelength $\boldsymbol{\alpha}$ related to the minimal strain energy, is presented by the straight red lines.

In previous studies *approximate* analytic solutions were obtained by minimizing the elastic energy of the system (e.g., [3]). Although these approximate solutions resemble (9) they do not satisfy the equilibrium equation as was shown here.

2.3 Nonlinear foundation

When the displacements in a buckled layer become sufficiently large relative to the thickness of the elastic foundation, the foundation can no longer be modeled as linear (due to the physical inconsistency that the thickness of a linear foundation can be reduced to zero by applying a finite force). A more realistic model of elastic foundation should include of stiffening in compression and possible softening in extension (e.g., [3]).

The effect of stiffening and softening elastic foundations on the *postbuckling* behavior of the system is investigated. To this end, the governing equation (1) is rewritten, for stiffening or softening foundation, and is numerically solved by means of the finite differences implemented in a MATLAB code. The non-linear softening and stiffening elastic foundations are modeled here by replacing the last term ($\tilde{y}$) in (1) by one of the following terms:

$$\tilde{b}\sinh^{-1}\left(\tilde{y}/\tilde{b}\right) \qquad \text{(softening foundation)} \qquad (10a)$$

$$\tilde{b}\sinh\left(\tilde{y}/\tilde{b}\right) \qquad \text{(stiffening foundation)} \qquad (10b)$$

Here $\tilde{b}$ is a dimensionless parameter that determines the intensity of the stiffening and softening elastic foundations. In case of small deformation, the above non-linear foundations asymptotically approach the linear foundation.

Figure 4 presents the normalized wavelength α as function of the normalized pre-stress β for a stiffening, linear, and softening foundations. α decreases with increasing β for a stiffening foundation, is constant for a linear foundation, and increases with increasing β for a softening foundation.

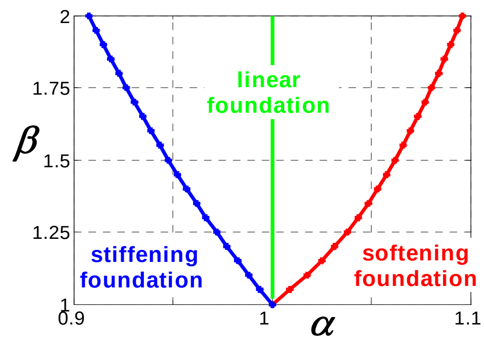

Figure 4: Post-buckling pre-stress $\boldsymbol{\beta}$ vs. wavelength $\boldsymbol{\alpha}$ for a linear, stiffening and softening elastic foundations.

3 ELECROMECHANICAL BUCKLING

Mechanical buckling due to internal compressive stress is one kind of instability. Another type of instability that is prevalent in MEMS is the well known electromechanical pull-in phenomenon.

In this work we investigate the instability that arises when a thin layer is subjected to *both* a compressive pre-stress and electrostatic forces. The electrostatic forces are induced by applying a voltage difference between the conductive pre-stressed layer and the conductive substrate (Fig. 1). This force takes the form

$$\tilde{q} = \frac{1}{2}\frac{\varepsilon_0 V^2}{(g-y)^2}\frac{\sqrt{S}}{\Lambda_{cr} k_f} \qquad (10)$$

where g is the foundation thickness.

In this section we are interested in the affect of the electrostatic forces on the critical state. Accordingly, the term in square brackets in (1) is omitted, and the equation is re-normalized in the form

$$\frac{1}{(2\pi)^4}\frac{d^4\bar{y}}{d\tilde{x}^4} + 2\beta\frac{1}{(2\pi)^2}\frac{d^2\bar{y}}{d\tilde{x}^2} + \bar{y} = \frac{\bar{V}^2}{(1-\bar{y})^2} \qquad (11)$$

where

$$\bar{y} = \frac{\tilde{y}}{\tilde{g}}, \quad \tilde{g} = \frac{\sqrt{S}}{\Lambda_{cr}}g, \quad \bar{V}^2 = \frac{\varepsilon b V^2 \Lambda_{cr} S^{3/2}}{2E^* I\tilde{g}^3(2\pi)^4}$$

Here we postulate a deflection in the form

$$\bar{y} = \bar{y}_0 + B\sin\left(2\pi\tilde{x}/\alpha\right) \tag{12}$$

where $\bar{y}_0$ is the average electromechanical displacement, and B is the amplitude of the structural waves that develop due to buckling. If no buckling occurs, the problem reduces to the case of a parallel-plates electrostatic actuator. At the verge of buckling, B is small, and the electrostatic force is approximated by a Taylor expansion. Substituting the postulated solution (12) and the first two terms of the Taylor expansion into (11) yields

$$B\frac{\sin\left(2\pi\tilde{x}/\alpha\right)}{\alpha^4}\left[\alpha^4(1-\delta)-2\alpha^2\beta+1\right]=\frac{\bar{V}^2}{\left(1-\bar{y}_0\right)^2}-\bar{y}_0 \tag{13}$$

where δ is the normalized electrostatic stiffness given by

$$\delta=\frac{2\bar{V}^2}{\left(1-\bar{y}_0\right)^3} \tag{14}$$

On the verge of buckling (i.e., $B = 0$), the trivial solution is

$$\bar{V}^2=\bar{y}_0\left(1-\bar{y}_0\right)^2 \tag{15}$$

The deflection of the layer in this case is uniform, $\bar{y}=\bar{y}_0$. For incipient buckling, (15) holds and can be subtracted from (13) to yield the electromechanical buckling equation

$$\alpha^4(1-\delta)-2\alpha^2\beta+1=0 \tag{16}$$

The solution of the last equation is

$$\alpha=\sqrt{\frac{\beta\pm\sqrt{\beta^2-(1-\delta)}}{1-\delta}} \tag{17}$$

At the critical state $\beta_{cr}=\sqrt{1-\delta}$, and consequently the critical value of the wavelength parameter is found to be

$$\alpha_{cr}=1\Big/\sqrt{\beta_{cr}} \tag{18}$$

In the previous discussion it was shown that in the absence of electrostatic forces buckling cannot occur for β<1. In contrast, from (17) it is clear that due to the destabilizing affect of the electrostatic forces, buckling can occur for β<1 provided that δ>0.

For a given normalized pre-stress β, the critical voltage $\bar{V}_{cr}$ that switches the flat layer into its buckled state is extracted by solving (14) with (15) to yield

$$\bar{V}_{cr}{}^2=4\frac{1-\beta_{cr}{}^2}{\left(3-\beta_{cr}{}^2\right)^3} \tag{19}$$

Figures 5 and 6 present the normalized voltage $\bar{V}_{cr}{}^2$ and normalized wavelength α_{cr}, at the critical states, for various values of normalized pre-stress β. At the limit of zero pre-stress, the critical wavelength is infinite and the voltage approaches an asymptotic value. This state is in essence the pull-in state of an infinite parallel-plates actuator.

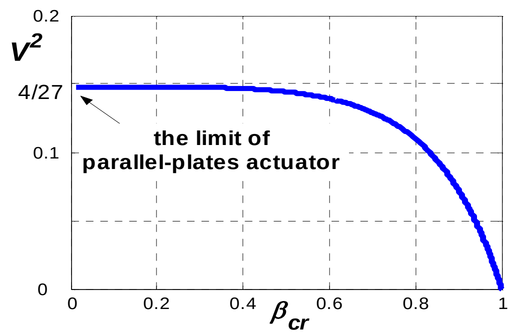

Figure 5: The normalized critical voltage $\bar{V}^2$ and critical pre-stress $\boldsymbol{\beta_{cr}}$ of electromechanical buckling. At zero pre-stress the normalized critical voltage is the same as for a parallel-plates actuator.

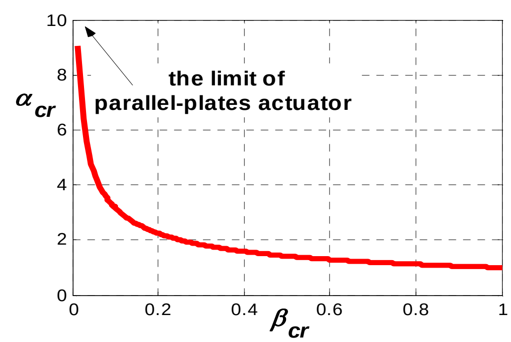

Figure 6: The normalized critical wavelength $\boldsymbol{\alpha_{cr}}$ and critical pre-stress $\boldsymbol{\beta_{cr}}$ of electromechanical buckling. At zero pre-stress the normalized wavelength is infinite.

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