The Application of Parametric Excitation to a Micro-Ring Gyroscope

B.J. Gallacher, J.S. Burdess, A.J. Harris and J. Hedley
Institute for Nanoscale Science and Technology, University of Newcastle upon Tyne
b.j.gallacher@ncl.ac.uk

ABSTRACT

A parametric excitation scheme that separates the drive and response frequencies of an electrostatically driven ring gyroscope in the approximate ratio of 10:1 is reported. This frequency separation permits reduction of troublesome electrical feedthrough common in many MEMS devices.

Keywords: parametric, excitation, gyroscope, feedthrough

1 INTRODUCTION

This paper reports on a parametric excitation scheme that separates the drive and response frequencies of a micro-ring gyroscope in the ratio 10:1 approximately, with the aim of minimising electrical "feedthrough". Electrical feedthrough of the drive signal due to parasitic capacitance is a common problem in many MEMS devices actuated electrostatically, inductively and piezoelectrically and in the case of the gyroscope, limits its sensitivity to applied angular velocities. To date research into the application of parametric excitation to MEMS/NEMS has been restricted to subharmonic and superharmonic parametric resonances [1,2,3,4]. In this case the ratio of the drive and response frequencies has a maximum value of two, for first-order parametric resonance. For a multi-dimensional system, combination resonances are excited when the frequency of modulation of a system parameter is a multiple of either the sum or difference of two of the natural frequencies of the system [5]. By ensuring the drive frequency is the sum of the natural frequency of a higher order mode and the natural frequency of the mode required to be excited, the ratio of the drive to response frequencies will be greater than two. In this paper, sum combination resonances between the 2nd and 5th order in-plane flexural modes are considered and are shown to be an alternative method of excitation suitable for the gyroscope.

2 EQUATION OF MOTION

Figure (1) illustrates the ring gyroscope consisting of planar ring of radius *a*, width *b* and thickness *d*. Actuation and sensing is performed electrostatically using a plurality of electrodes also of thickness *d*, displaced radially from the outer surface of the ring by a distance h_o. The parametrically excited ring gyroscope takes advantage of the dynamic instability possible when the condition for a sum combination resonance is met. Therefore, the dynamics of the ring will be expressed in terms of two pairs of orthogonal flexural modes of order n and m, where m<n.

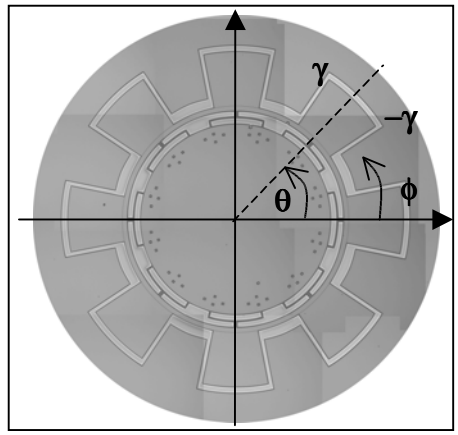

Figure (1) Photograph of ring gyroscope

Excitation of the ring structure into vibration is accomplished electrostatically through a plurality of *p* cyclically arranged electrodes. The electrical energy stored in the series of capacitors, formed between the ring (assumed to be held at Earth potential) and the electrodes, each biased with a voltage U_i is given by

$$E_E = \frac{\varepsilon_o a d}{2h_o}\sum_{i=1}^{p} U_i^{\,2}\int_{-\gamma+\theta_i}^{+\gamma+\vartheta_i}\left[1+\frac{u}{h_o}+\left(\frac{u}{h_o}\right)^2+\left(\frac{u}{h_o}\right)^3+\cdots\right]d\phi \quad (1)$$

The radial displacement *u,* of a point on the centre line of the ring may be expressed in terms of the undamped, unforced flexural modes of an unsupported ring by:

$$u = q_1\cos n\phi + q_2\sin n\phi + q_3\cos m\phi + q_4\sin m\phi \quad (2)$$

where q_i and are generalised coordinates which represent the contribution made by each mode to the motion.

The radial displacement of the ring is assumed to be small compared to the nominal air gap separation thus terms of order greater than $(u/h_o)^3$ may be ignored. The generalised stiffness matrix and forcing vector may be obtained from substituting in the expression for the radial displacement given by equation (2) into equation (1) and by determining $\frac{\partial E_E}{\partial q_i}$ and $\frac{\partial^2 E_E}{\partial q_i^{\,2}}$ for i=1..4.

The equation of motion of the ring excited electrostatically by a voltage $U(t)$ is given by

$$[m]\underline{q}+[c]\underline{q}+[k]\underline{q}=\frac{\varepsilon_o ad}{h_o^{\,3}}U(t)^2[K]\underline{q}(t)+\frac{\varepsilon_o ad}{2h_o^{\,2}}U(t)^2\,\underline{G}$$

where

$$[m]=\begin{bmatrix} m_n & & & \\ & m_n & & \\ & & m_m & \\ & & & m_m \end{bmatrix},\quad [k]=\begin{bmatrix} k_n & & & \\ & k_n & & \\ & & k_m & \\ & & & k_m \end{bmatrix}$$

$$[c]=\begin{bmatrix} 2m_n\xi_1\omega_1 & & & \\ & 2m_n\xi_2\omega_2 & & \\ & & 2m_m\xi_3\omega_3 & \\ & & & 2m_m\xi_4\omega_4 \end{bmatrix}$$

and

$$m_n=m_r\left(1+\frac{1}{n^2}\right),\ m_m=m_r\left(1+\frac{1}{m^2}\right),\ m_r=\pi\rho abd$$

$$k_n=\pi\left(1-n^2\right)^2\frac{\mathrm{EI}}{a^3}\ ,\ k_m=\pi\left(1-m^2\right)^2\frac{\mathrm{EI}}{a^3}$$

The matrix $[K]$ and vector $\underline{G}$ are the spatial parts of the generalised stiffness and forcing vector corresponding to the modal vector $\underline{q}=(q_1,\ q_2, q_3, q_4)^{\mathrm{T}}$.

It is convenient to introduce the non-dimensional parameter μ such that $\mu=U(t)^2/U_p^{\,2}$ where U_p is the pull-in voltage. For typical drive voltages $\mu<<1$ and the equation of motion may be rewritten in the form

$$\underline{q}+[2\xi\omega]\underline{q}+\left\{\left[\omega^2\right]-[D]\mu\right\}\underline{q}=\mu\underline{F}$$

where

$$[\mathrm{D}]=\frac{\varepsilon_o ad}{h_o^{\,3}}U_p^{\,2}[m]^{-1}[K]\ \text{and}\ \underline{F}=\frac{\varepsilon_o ad}{2h_o^{\,2}}U_p^{\,2}\,\underline{G}$$

The periodic voltage applied to ring may be conveniently represented by the Fourier series

$$\mu=\varepsilon\eta\left(u_o+\sum_{r=1}^{\infty}u_r e^{ir\omega t}+\sum_{r=1}^{\infty}u_r^{\,*}e^{-ir\omega t}\right)$$

where $\varepsilon<<1$ and η represents the voltage scaling factor.

The equation of motion is therefore given by

$$q_i+2\varepsilon\upsilon_i\omega_i q_i+\left\{\omega_i^{\,2}q_i-\varepsilon\eta u_o\sum_{j=1}^{4}D_{ij}q_j\right\}-$$
$$\varepsilon\eta\left(\sum_{r=1}^{\infty}u_r e^{ir\omega t}+\sum_{r=1}^{\infty}u_r^{\,*}e^{-ir\omega t}\right)\sum_{j=1}^{4}D_{ij}q_j=$$
$$\varepsilon\eta\left(u_o+\sum_{r=1}^{\infty}u_r e^{ir\omega t}+\sum_{r=1}^{\infty}u_r^{\,*}e^{-ir\omega t}\right)F_i \qquad (3)$$

where use was made of the relation $\zeta_i=\varepsilon\upsilon_i$ since operation is under vacuum. Equation (3) represents a set of coupled Hill's equations. As the flexural modes of vibration of the perfect ring occur in degenerate pairs, the natural frequencies of a slightly imperfect ring may be expressed in the form [6]

$$\omega_i=\omega_\alpha+\varepsilon\alpha_i\quad i=1,2.$$
$$\omega_k=\omega_\beta+\varepsilon\beta_k\quad k=3,4. \qquad (4)$$

where α and β represent mis-tuning due to structural imperfections and $\alpha_1=\beta_3=0$.

3 PERTURBATION ANALYSIS

A multiple time scales perturbation analysis is performed [7] using ε as the perturbation parameter to investigate the dynamic stability and mode coupling. To determine the stability boundaries it is necessary to rewrite the condition for the sum resonance as

$$r\omega=\omega_\alpha+\omega_\beta+\varepsilon r\lambda$$

where λ represents slight mis-tuning from the sum resonance condition. Using the relations given by equation (4), this can be written as

$$r\omega=\omega_i+\omega_k+\varepsilon\left[r\lambda-\left(\alpha_i+\beta_k\right)\right]$$

In the multiple time scales method the solution is expressed as an asymptotic expansion

$$q_i\left(\bar{t},\hat{t}\right)=q_i^{(0)}+\varepsilon q_i^{(1)}+\ \cdots+O\left(\varepsilon^2\right) \qquad (5)$$

where $\hat{t}=t+\cdots O\left(\varepsilon^2\right)$ and $\bar{t}=\varepsilon t$. (6)

As the generalised coordinates q_i are functions of both the slow and fast time scales $\bar{t}$ and $\hat{t}$, the time derivatives are given by

$$q_i=\frac{\partial q_i^{(0)}}{\partial\hat{t}}+\varepsilon\frac{\partial q_i^{(1)}}{\partial\hat{t}}+\varepsilon\frac{\partial q_i^{(0)}}{\partial\bar{t}}+\ \cdots+\mathrm{O}\left(\varepsilon^2\right) \qquad (7)$$

$$q_i=\frac{\partial q_i^{(0)}}{\partial\hat{t}}+\varepsilon\frac{\partial^2 q_i^{(1)}}{\partial\hat{t}}+2\varepsilon\frac{\partial^2 q_i^{(0)}}{\partial\bar{t}\partial\hat{t}}+\ \cdots+\mathrm{O}\left(\varepsilon^2\right) \qquad (8)$$

Using of the equations (5,7 and 8) and equating powers of ε, equation (3) yields the recurrent equations

$$\frac{\partial^2 q_i^{(0)}}{\partial\hat{t}^{\,2}}+\omega_i^{\,2}q_i^{(0)}=0 \qquad (9)$$

$$\frac{\partial^2 q_i^{(1)}}{\partial\hat{t}^{\,2}}+\omega_i^{\,2}q_i^{(1)}=-2\frac{\partial^2 q_i^{(o)}}{\partial\hat{t}\bar{t}}-2\nu_i\omega_i\frac{\partial q_i^{(o)}}{\partial\hat{t}}+$$
$$\eta\left(u_o+\sum_{r=1}^{\infty}u_r e^{ir\omega\hat{t}}+\sum_{r=1}^{\infty}u_r^{\,*}e^{-ir\omega\hat{t}}\right)\sum_{j=1}^{4}D_{ij}q_j^{(o)}+$$
$$\eta\left(u_o+\sum_{r=1}^{\infty}u_r e^{ir\omega\hat{t}}+\sum_{r=1}^{\infty}u_r^{\,*}e^{-ir\omega\hat{t}}\right)F_i \qquad (10)$$

The general solution to equation (9) is

$$q_i^{(0)} = A_{oi}(\bar{t})e^{i\omega_i\hat{t}} + B_{oi}(\bar{t})e^{-i\omega_i\hat{t}}. \tag{11}$$

A_{oi} and B_{oi} are determined by imposing the condition that the solution to equation (10) is uniform in $\hat{t}$. Equation (10) represents a system subjected to both external and parametric forcing through the F_i and D_{ij} terms, respectively. As any resonance producing terms in the equation of motion will lead to unbounded particular solutions with respect to $\hat{t}$, these must be removed to meet the condition for uniform solutions.

Parametric resonance occurs when $j=i$ or $j=k$.
Thus conditions for solutions uniform in $\hat{t}$ are given by

$$\frac{\partial A_{oi}}{\partial \bar{t}} + \nu_i\omega_i A_{oi} + i\eta u_o \frac{D_{ii}}{2\omega_i} A_{oi} + i\eta u_r \frac{D_{ik}}{2\omega_i} B_{ok} e^{i\bar{\lambda}\bar{t}} = 0$$

$$\frac{\partial B_{oi}}{\partial \bar{t}} + \nu_i\omega_i B_{oi} - i\eta u_o \frac{D_{ii}}{2\omega_i} B_{oi} - i\eta u_r^* \frac{D_{ik}}{2\omega_i} A_{ok} e^{-i\bar{\lambda}\bar{t}} = 0$$

$$\frac{\partial A_{ok}}{\partial \bar{t}} + \nu_k\omega_k A_{ok} + i\eta u_o \frac{D_{kk}}{2\omega_k} A_{ok} + i\eta u_r \frac{D_{ki}}{2\omega_k} B_{oi} e^{i\bar{\lambda}\bar{t}} = 0$$

$$\frac{\partial B_{ok}}{\partial \bar{t}} + \nu_k\omega_k B_{ok} - i\eta u_o \frac{D_{kk}}{2\omega_k} B_{ok} - i\eta u_r^* \frac{D_{ki}}{2\omega_k} A_{oi} e^{-i\bar{\lambda}\bar{t}} = 0 \tag{12}$$

where $\bar{\lambda} = r\lambda - (\alpha_i + \beta_k)$ and use was made of the relation $\bar{t} = \varepsilon t$.
The solutions to the above set of equations are

$$A_{oi} = A_{oi}^{(o)} e^{\left(\sigma + i\frac{\bar{\lambda}}{2}\right)\bar{t}},\ A_{ok} = A_{ok}^{(o)} e^{\left(\sigma + i\frac{\bar{\lambda}}{2}\right)\bar{t}},$$

$$B_{oi} = B_{oi}^{(o)} e^{\left(\sigma - i\frac{\bar{\lambda}}{2}\right)\bar{t}} \text{ and } B_{ok} = B_{ok}^{(o)} e^{\left(\sigma - i\frac{\bar{\lambda}}{2}\right)\bar{t}} \tag{13}$$

Substituting the solutions given by equation (13) into equation (12) yields the quadratic

$$\sigma^2 + (\nu_i\omega_i + \nu_k\omega_k)\sigma + \nu_i\omega_i\nu_k\omega_k + \frac{\bar{\lambda}^2}{4} + \frac{\bar{\lambda}}{4}\eta u_o\varepsilon\left(\frac{D_{ii}}{\omega_i} + \frac{D_{kk}}{\omega_k}\right) + \frac{\eta^2 u_o^2\varepsilon^2}{4}\frac{D_{ii}}{\omega_i}\frac{D_{kk}}{\omega_k} - \frac{\eta^2 u_r u_r^* \varepsilon^2}{4}\frac{D_{ik}}{\omega_i}\frac{D_{ki}}{\omega_k} = 0$$

Clearly for stable solutions the real part of σ must be zero. This condition is met when λ is described by

$$r\lambda = -\frac{1}{2}\left[\eta u_o\left(\frac{D_{ii}}{\omega_i} + \frac{D_{kk}}{\omega_k}\right) \pm \sqrt{\eta^2 u_o^2\left(\frac{D_{ii}}{\omega_i} - \frac{D_{kk}}{\omega_k}\right)^2 + 4\eta^2|u_r|^2\frac{D_{ik}}{\omega_i}\frac{D_{ki}}{\omega_k} - 16\nu_i\omega_i\nu_k\omega_k}\right] + (\alpha_i + \beta_k) \tag{14}$$

4 EXCITATION FROM A SINGLE ELECTRODE

It can be shown that for any single drive electrode, the matrix D will be in the form

$$[D] = \begin{bmatrix} D_{11} & 0 & D_{13} & 0 \\ 0 & D_{22} & 0 & D_{24} \\ \frac{m_m}{m_n}D_{13} & 0 & D_{33} & 0 \\ 0 & \frac{m_m}{m_n}D_{24} & 0 & D_{44} \end{bmatrix} \tag{15}$$

Therefore, the modes corresponding to coordinates q_1,q_3 and q_2,q_4 are parametrically coupled through the matrix elements D_{13} and D_{24}, respectively. For the perfect axisymmetric ring $\omega_i=\omega_\alpha$ and $\omega_k=\omega_\beta$ for i=1,2 and k=3,4. Therefore, when the condition for a sum resonance between the n and m-order modes is met, the relative magnitudes of D_{13}, D_{24} determine which mode pair is the first excited when the critical voltage is exceeded. It may be shown that the mode pair q_1,q_3 is excited first when $\gamma<\pi/(n+m)$. The ring gyroscope considered makes use of the n=2 modes to sense rate of rotation. Hence eight cyclically arranged electrodes with $\gamma = \pi/16$ radians provide the drive and sense functions. Thus the mode pair q_1,q_3 will be excited first for values of m up to a maximum value of 13.

5 STABILITY BOUNDARIES

Consider the case where excitation is provided by a single electrode assumed to be aligned with the nodal diameter of the sin(nθ) mode. A square wave voltage waveform with Fourier components $u_o = 1/2$ and $|u_1| = 1/\pi$ provides the excitation. The modes utilised in the case have orders n=2 and m=5 and their mode shapes are shown in figure (2).

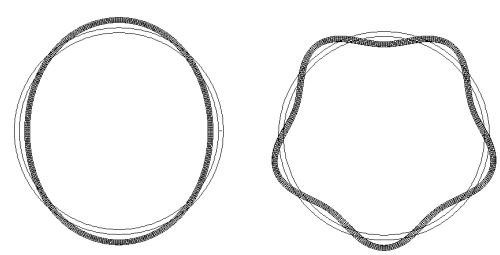

Figure (2) cos2ϕ and cos5ϕ mode shapes

The damping ratios of the n and m- order modes are assumed to be equal and have the value 2.5 x 10^{-5}. This is typical of the measured damping ratio of vacuum-packed devices for in-plane flexural modes. It is assumed that the ring is slightly imperfect with $\alpha_1=\beta_3=0$, $\varepsilon\alpha_2=0.0001\omega_2$ and $\varepsilon\beta_4=0.0001\omega_4$. This constitutes a frequency split of 0.1%

between the otherwise degenerate modes and is similar to that measured in fabricated devices.

Using equation (14) the stability boundaries may be drawn. There are two stability boundaries for each order of mode. The two boundaries are displaced with respect to each other due to the mis-tuning between the otherwise degenerate modes and also to the asymmetric form of the electrostatic stiffness matrix when one drive electrode is used. As a result, the onset of instability in the q_1,q_3 mode pair occurs at a lower excitation voltage than the q_2,q_4 mode pair. Therefore, it will be possible to excite only the q_1,q_3 mode pair into oscillation. The minimum voltage amplitude required to induce instability in the q_1,q_3 mode pair is shown on Figure (3) and has the value $\eta_m=162$. This value occurs at a excitation frequency described by $\omega = \omega_1^* + \omega_3^*$, where ω_1^* and ω_3^* are the damped natural frequencies of the cos(nϕ) and cos(mϕ) modes, respectively. When n=2 and m=3, $\omega \approx 10\omega_1^*$.

6 CONCLUSIONS

It has been shown that an electrostatically actuated ring gyroscope may be excited using combination parametric resonances between the in-plane flexural modes of order n=2 and m=5. This frequency separates the drive and response signals by approximately a decade and enables effective filtering of the electrical feedthrough of the drive signal which typically contaminates the sense signal produced in response to a rate of rotation and limits the device performance.

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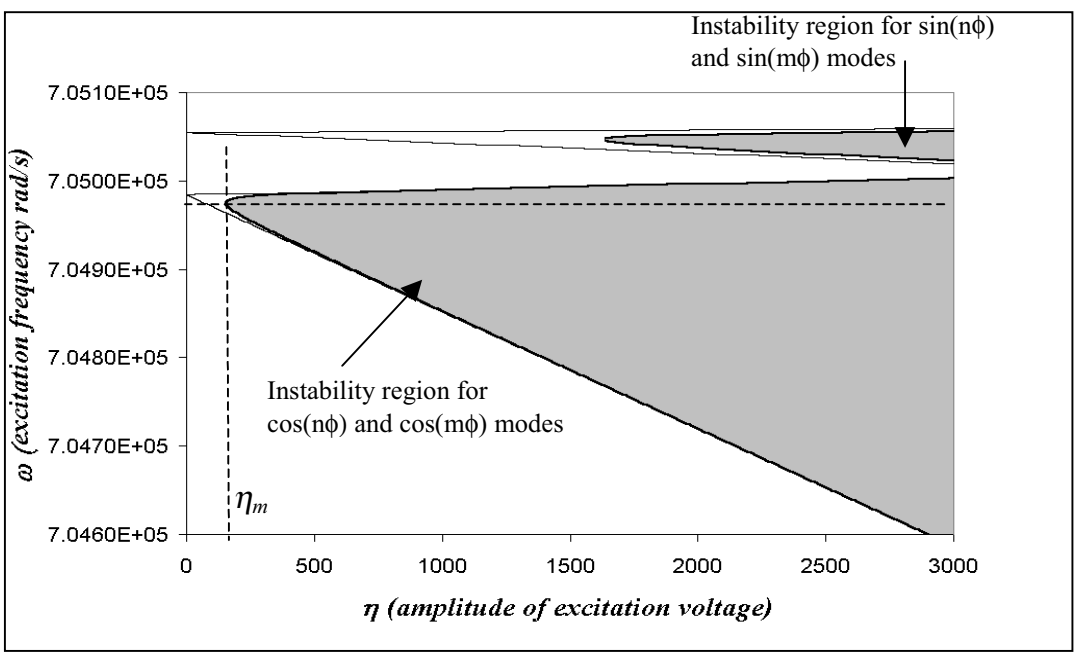

Figure (3) Stability map for n =2, m =5, one drive electrode